Journal of Engineering Physics and Thermophysics, Vol. 81, No. 1, 2008

A CLASS OF INTEGRAL TRANSFORMATIONS FOR THE GENERALIZED EQUATION OF NONSTATIONARY HEAT CONDUCTION

É. M. Kartashov UDC 536.2.001

Integral transformations for finding analytical solutions of the boundary-value problems of nonstationary heat conduction for the generalized equation of transfer in an infinite region bounded on the inside by a plane, cylindrical, or spherical surface have been constructed.

The method of integral transformations developed by A. V. Luikov in [1] in solving boundary-value problems of nonstationary heat conduction in the regions of canonical type (plate, cylinder, sphere) has in subsequent years undergone a further development with respect to its theory [2, 3] and application [4–6]. The indicated method is indispensable in finding solutions of the classical linear boundary-value problems of transfer with inhomogeneities of general type in both the basic equation and boundary-value conditions and in the presence of basic functional relations of the method — determination of the transformation, reversion formula, image of the Laplace operator — all calculations are reduced to simple algorithmic transformations. Despite the results achieved in this field, a number of problems remain open and need further consideration. One of them is the construction of integral transformations for the generalized equation of nonstationary heat conduction in an infinite region bounded on the inside by a plane, cylindrical, or spherical surface. We speak of an equation of the form

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{2m+1}{x}\frac{\partial T}{\partial x}\right) \tag{1}$$

in the region with $G = \{x > x_0\}$, $t > 0$ on the boundary of which one of the following conditions is prescribed: temperature heating

$$T(x, t)\big|_{x=x_0} = \varphi(t), \quad t > 0, \tag{2}$$

thermal heating

$$\left.\frac{\partial T(x, t)}{\partial x}\right|_{x=x_0} = (1/\lambda_t)\,\varphi(t), \quad t > 0, \tag{3}$$

heating by a medium

$$\left.\frac{\partial T(x, t)}{\partial x}\right|_{x=x_0} = h\left[T(x, t)\big|_{x=x_0} - \varphi(t)\right], \quad t > 0, \tag{4}$$

as well as

$$\left|T(x, t)\right| < +\infty, \quad x \ge x_0, \quad t \ge 0. \tag{5}$$

M. V. Lomonosov Moscow State Academy of Fine Chemical Technology, 86 Vernadskii Ave., Moscow, 119571, Russia; email: kartashov@mitht.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 81, No. 1, pp. 123–130, January–February, 2008. Original article submitted May 15, 2007.

1062-0125/08/8101-0129©2008 Springer Science+Business Media, Inc.

At $m = 0$ the region is bounded on the inside by a cylindrical surface, at $m = 1/2$ — by a spherical one, and at $m = -1/2$ — by a plane one. As will be shown below, the developed approach admits in (1), on the right, the presence of the heat source function $F(x, t)/(c\rho)$, inhomogeneous initial condition $T(x, 0) = \Phi_0(x)$, $x \geq x_0$, as well as of the dependences of the thermal diffusivity and thermal conductivity coefficients on time $a = a(t)$, $\lambda = \lambda_t(t)$, where $a(t)$, $\lambda_t(t)$ are the nonnegative continuous functions in the region $t \in [0, \infty)$.

In further considerations we will use the theory of the singular Sturm–Liouville problem over a semi-infinite stretch [2, 3], the foundations of which will be considered in application to (1)–(4), thus extending the theory of [2, 3] to the cases studied.

Let $L \equiv [q(x) - d^2/dx^2]$ be the linear operator applied to the function $v(x)$, where $q(x)$ is the given continuous function in the region $G = [x_0, \infty)$. Let us denote by $L^{(2)}[x_0, \infty)$ the class of functions with the square of the module integrated over $[x_0, \infty)$. We will consider the problem

$$L\,[v\,(x)] = \lambda v\,(x, \lambda)\,, \quad x > x_0\,, \tag{6}$$

$$\alpha_1 v'\,(x_0, \lambda) - \beta_1 v\,(x_0, \lambda) = 0\,. \tag{7}$$

Let λ be a certain fixed complex number which is not a real one. The operator L has the following important property: if all the solutions of Eq. (6) belong to the class $L^{(2)}[x_0, \infty)$ at a certain complex value of λ, then they belong to that class also at all complex λ values.

Let $\psi(x, \lambda)$ and $\chi(x, \lambda)$ be the solutions of Eq. (6) with the boundary conditions

$$\psi\,(x_0, \lambda) = \alpha_1\,, \quad \psi'\,(x_0, \lambda) = \beta_1\,, \tag{8}$$

$$\chi\,(x_0, \lambda) = \alpha_2\,, \quad \chi'\,(x_0, \lambda) = -\,\beta_2\,, \tag{9}$$

that satisfy the equality

$$\alpha_1\beta_2 + \alpha_2\beta_1 = 1\,, \quad \alpha_i^2 + \beta_i^2 > 0\,, \quad i = 1, 2\,. \tag{10}$$

We note that condition (10) in the approach expounded is of fundamental importance. The solutions ψ and χ are linearly independent, since their Wronskian at the point x_0 is equal to

$$W\,(\chi, \psi)\,\big|_{x=x_0} = 1 \tag{11}$$

and therefore differs from zero on the entire interval $[x_0, \infty)$. In this case, the function $\psi(x, \lambda)$ satisfies the boundary condition (7) for Eq. (1), i.e.,

$$\alpha_1 \psi'\,(x_0, \lambda) - \beta_1 \psi\,(x_0, \lambda) = 0\,, \tag{12}$$

so that any solution $v(x, \lambda)$ of Eq. (1) that satisfies condition (2) differs from $\psi(x, \lambda)$ only by a numerical factor. From the linear independence of χ and ψ it also follows that accurate to within the factor any solution of Eq. (1) different from ψ can be represented in the form

$$v\,(x, \lambda) + \chi\,(x, \lambda) + m_\infty\,(\lambda)\,\psi\,(x, \lambda)\,, \tag{13}$$

where $m_\infty(\lambda)$ is a certain complex number which plays an important role in the theory of construction of the unknown integral transformations on the basis of the expansion in eigenfunctions of the singular Sturm–Liouville problem on the semi-infinite interval. We will formulate the basic theory of expansion. Let $f(x) \in L^{(2)}[x_0, \infty)$ be continuous and there exist the following function:

$$\bar{f}(\lambda) = \int_{x_0}^{\infty} f(x)\, \psi(x, \lambda)\, dx \,, \tag{14}$$

where $\psi(x, \lambda)$ is the solution of Eq. (6) with boundary conditions (8). Then

$$f(x) = \int_{-\infty}^{+\infty} \bar{f}(\lambda)\, \psi(x, \lambda)\, d\sigma(\lambda) \,, \tag{15}$$

where integration is carried out along the real axis. It is assumed that the integral in (15) converges absolutely and on each finite interval uniformly over x. Relations (14) and (15) establish the possibility of expansion of any function $f(x) \in L^{(2)}[x_0, \infty)$ into eigenfunctions of the singular Sturm–Liouville problem on a semi-infinite interval. They can be written in the form

$$f(x) = \int_{-\infty}^{+\infty} \psi(x, \lambda)\, d\sigma(\lambda) \int_{x_0}^{\infty} \psi(\xi, \lambda) f(\xi)\, d\xi \,. \tag{16}$$

The functions $\sigma(\lambda)$ and $m_\infty(\lambda)$ (according to the terminology adopted, these are the spectral and limiting function, respectively [2, 3]) are closely connected: one can yield the other:

$$d\sigma(\lambda) = \frac{1}{\pi} \operatorname{Im}[m_\infty(\lambda)]\, d\lambda \,. \tag{17}$$

In the presence of an infinitely removed point in the G region the value of $m_\infty(\lambda)$ is determined by an unique technique based on the existence of the single linearly independent solution of Eq. (6) belonging to the class $L^{(2)}[x_0, \infty)$. Indeed, as shown below, the spectral problem for (1)–(5) leads to the case where $\psi(x, \lambda) \notin L^{(2)}[x_0, \infty)$; however, when $\operatorname{Im}(\lambda) \neq 0$, there exists the solution of Eq. (6) of class $L^{(2)}[x_0, \infty)$ that is linearly independent of ψ, namely,

$$\chi(x, \lambda) + m_\infty(\lambda)\, \psi(x, \lambda) = v(x, \lambda) \in L^{(2)}[x_0, \infty) \,. \tag{18}$$

Two linearly independent solutions of class $L^{(2)}[x_0, \infty)$ cannot exist, because then all the solutions of Eq. (6) would belong to $L^{(2)}[x_0, \infty)$, which contradicts the statement that $\psi(x, \lambda) \notin L^{(2)}[x_0, \infty)$. Thus, in the considered region G Eq. (6) at $\operatorname{Im}(\lambda) \neq 0$ has one and only one linearly independent solution of class $L^{(2)}[x_0, \infty)$, namely, $v(x, \lambda) = \chi(x, \lambda) + m_\infty(\lambda)\psi(x, \lambda)$. The latter means that all the solutions of Eq. (6) occurring in $L^{(2)}[x_0, \infty)$ are proportional to Eq. (18). This fact is used further to find the function $m_\infty(\lambda)$.

We will consider in more detail the procedure of constructing an integral transformation for case (2) in solving the first boundary-value problem of nonstationary heat conduction for Eq. (1) (with possible additional inhomogeneities in the initial statement of the problem).

The corresponding spectral problem has the form

$$\frac{d^2\Theta}{dx^2} + \frac{2m+1}{x}\frac{d\Theta}{dx} + s^2\Theta = 0 \,, \quad x > x_0 \,, \tag{19}$$

$$\Theta(x, s)\big|_{x=x_0} = 0 \,, \tag{20}$$

$$\left|\Theta(x, s)\right| < +\infty \,, \quad x \geq x_0 \,. \tag{21}$$

With the aid of the substitution of

$$\Theta(x, s) = x^{-(m+1/2)} v(x, \lambda) \tag{22}$$

Eq. (19) is reduced to the form of Eq. (6):

$$\frac{d^2 v}{dx^2} + \left(\lambda - \frac{m^2 - 1/4}{x^2}\right) v = 0, \quad \lambda = s^2, \quad x > x_0, \tag{23}$$

with the boundary condition

$$v(x_0, \lambda) = 0, \tag{24}$$

whence, according to Eq. (7),

$$\alpha_1 = 0, \quad \beta_1 = 1. \tag{25}$$

The expressions $\sqrt{x}\, J_m(sx)$ and $\sqrt{x}\, Y_m(sx)$ are linearly independent solutions of Eq. (23), where s is one of the values of $\sqrt{\lambda}$ (in what follows s means the value for which $-\pi/2 < \arg s < \pi/2$). The functions $\chi(x, \lambda)$ and $\psi(x, \lambda)$ considered above — the solutions of Eq. (23) — must satisfy the conditions

$$\psi(x_0, \lambda) = 0, \quad \psi'(x_0, \lambda) = 1, \quad \chi(x_0, \lambda) = 1, \quad \chi'(x_0, \lambda) = 0$$

and must be expressed in terms of the fundamental system of the solutions of Eq. (23) in the following way:

$$\psi(x, \lambda) = -\frac{2}{\pi} \sqrt{x_0 x} \left[J_m(sx)\, Y_m(sx_0) - Y_m(sx)\, J_m(sx_0) \right], \tag{26}$$

$$\chi(x, \lambda) = \frac{2}{\pi} s \sqrt{x_0 x} \left[J_m(sx)\, Y'_m(sx_0) - Y_m(sx)\, J'_m(sx_0) \right] - \frac{\psi(x, \lambda)}{2x_0}. \tag{27}$$

It should be emphasized that the function $\psi(x, \lambda)$ is connected with the unknown function $\Theta(x, \lambda)$ — the kernel of the subsequent integral transformation — by relation (22):

$$\Theta(x, s) = x^{-(m+1/2)} \psi(x, \lambda). \tag{28}$$

We will calculate the function $m_\infty(\lambda)$. Along with the above-indicated fundamental solutions of Eq. (23), among the particular solutions of this equation there are also $\sqrt{x}\, H_m^{(1)}(sx)$ and $\sqrt{x}\, H_m^{(2)}(sx)$, where $H_m^{(1)}(z)$ and $H_m^{(2)}(z)$ are Hankel functions of the 1st and 2nd kind. From the asymptotic representations for the Hankel functions [2] it follows that when $\operatorname{Im} \lambda > 0$,

$$\sqrt{x}\, H_m^{(1)}(sx) \in L^{(2)}[x_0; \infty), \quad \sqrt{x}\, H_m^{(2)}(sx) \notin L^{(2)}[x_0; \infty). \tag{29}$$

Since in the region G for Eq. (23) only one linearly independent solution of the class $L^{(2)}[x_0, \infty)$ can exist that has the form of Eq. (18), and with account for Eq. (29), $\sqrt{x}\, H_m^{(1)}(sx)$ is also such a solution, $\nu(x, \lambda)$ may differ from $\sqrt{x}\, H_m^{(1)}(sx)$ only by the constant factor

$$\chi(x, \lambda) + m_\infty(\lambda)\, \psi(x, \lambda) = A\sqrt{x}\, H_\nu^{(1)}(sx) = A\sqrt{x} \left[J_m(sx) + iY_m(sx) \right], \tag{30}$$

where A is a constant. Having substituted expressionism (26) and (27) into Eq. (30) and equated the coefficients at $J_m(sx)$ and $Y_m(sx)$, we obtain

$$m_\infty(\lambda) = s\,\frac{\sqrt{x}\,H_m'^{(1)}(sx_0)}{\sqrt{x}\,H_m^{(1)}(sx_0)} + \frac{1}{2a}\,, \quad s = \sqrt{\lambda}\,. \tag{31}$$

Proceeding from the properties of the Bessel functions, we can see that $m_\infty(\lambda)$ is an even function of $\sqrt{\lambda}$, i.e., it is independent of the choice of the sign of $\sqrt{\lambda}$ and, consequently, is the single-values function $s = \sqrt{\lambda}$, where the value of $\sqrt{\lambda}$ is taken with a nonnegative real part. With the aid of Eq. (31) we can calculate Im $[m_\infty(\lambda)]$:

$$\mathrm{Im}\,[m_\infty(\lambda)] + \mathrm{Im}\left[s\,\frac{\sqrt{x}\,H_m'^{(1)}(sx_0)}{\sqrt{x}\,H_m^{(1)}(sx_0)}\right] = \mathrm{Im}\left[s\,\frac{J_m'(sx_0) + iY_m'(sx_0)}{J_m(sx_0) + iY_m(sx_0)}\right]. \tag{32}$$

After transformations with the use of the well-known properties of the Bessel functions, we find

$$\mathrm{Im}\,[m_\infty(\lambda)] = \frac{2}{\pi x_0}\,\frac{1}{J_m^2(sx_0) + Y_m^2(sx_0)}\,, \quad s, \lambda \ge 0\,, \quad s = \sqrt{\lambda}\,,$$

from which it follows that on the half-axis $\lambda \ge 0$ the function Im $[m_\infty(\lambda)]$ does not have singularities (the zeroes of the function $J_m(z)$ and those of $Y_m(z)$ do not coincide) and the spectral function $\sigma(\lambda)$ at $\lambda > 0$ is continuous. When $\lambda < 0$, the argument $sx_0 = x_0\sqrt{\lambda}$ is purely imaginary. Using the equality

$$H_m^{(1)}(z) = \frac{2}{\pi i}\exp\left[-\frac{\pi m}{2}\,i\right] K_m\left(\frac{z}{i}\right),$$

where $K_\nu(z)$ is the Macdonald function, we transform relation (32) as

$$\mathrm{Im}\,[m_\infty(\lambda)] = \mathrm{Im}\left[\sqrt{|\lambda|}\,\frac{K_m'(\sqrt{|\lambda|}\,x_0)}{K_m(\sqrt{|\lambda|}\,x_0)}\right], \quad \lambda < 0\,. \tag{33}$$

The Macdonald function $K_m(z)$ is real and positive at positive z, therefore the function on the right behind the symbol Im in Eq. (33) is real and does not have singularities, meaning that in (16) $d\sigma(\lambda) = 0$ when $\lambda < 0$. Thus, on the basis of Eqs. (17) and (33) we have

$$d\sigma(\lambda) = \frac{4}{\pi^2 x_0}\,\frac{s\,ds}{J_m^2(sx_0) + Y_m^2(sx_0)}\,, \quad \lambda = s^2 > 0\,. \tag{34}$$

Now, with account for relations (14)–(16), (26), (28), and (34), after simple transformations, we arrive at the sought integral transformation in solving the boundary-value problem of nonstationary heat conduction for Eq. (1) with boundary conditions of the first kind (2). We will write out successively all the necessary relations: the integral transformation of the function $T(x, t)$ in the region $x > x_0$:

$$\overline{T}(s, t) = \int_{x_0}^{\infty} \Theta(x, s)\,T(x, t)\,x^{2m+1}\,dx\,; \tag{35}$$

the kernel of the integral transformation

$$\Theta(x, s) = x^{-m}\,[J_m(sx)\,Y_m(sx_0) - Y_m(sx)\,J_m(sx_0)]\,; \tag{36}$$

the image of the operator

$$\Delta T(x,t) = \frac{\partial^2 T}{\partial x^2} + \frac{2m+1}{x}\frac{\partial T}{\partial x} = \frac{1}{x^{2m+1}}\frac{\partial}{\partial x}\left(x^{2m+1}\frac{\partial T}{\partial x}\right),$$

$$\int_{x_0}^{\infty} \Theta(x,s)\,\Delta T(x,t)\,x^{2m+1}\,dx = -\frac{2x_0^m}{\pi}\,T(x,t)\big|_{x=x_0} - s^2\bar{T}(x,t)\,; \tag{37}$$

the conversion formula

$$T(x,t) = \int_0^{\infty} \frac{\Theta(x,s)\,\bar{T}(s,t)}{J_m^2(sx_0) + Y_m^2(sx_0)}\,sds\,. \tag{38}$$

We will consider the following illustrative example. The solution of the problem

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \quad r > r_0\,, \quad t > 0\,, \tag{39}$$

$$T(r,t)\big|_{t=0} = 0\,, \quad r \ge r_0\,, \tag{40}$$

$$T(r,t)\big|_{r=r_0} = T_0\,, \quad t > 0\,, \quad |T(r,t)| < +\infty\,, \quad r \ge r_0\,, \quad t \ge 0\,, \tag{41}$$

obtained with the aid of Eqs. (35)–(38), at $m = 0$ has the form

$$T(r,t) = T_0 + \frac{2T_0}{\pi}\int_0^{\infty} \exp(-as^2 t)\,\frac{J_0(sr)\,Y_0(sr_0) - Y_0(sr)\,J_0(sr_0)}{s\,[J_0^2(sr_0) + Y_0^2(sr_0)]}\,ds\,.$$

The operational solution of this problem (according to Laplace):

$$\bar{T}(r,p) = T_0\,\frac{K_0\sqrt{p/(ar)}}{K_0\sqrt{p/(ar_0)}}$$

is associated with cumbersome calculations in transition to the inverted transform. For the boundary conditions of the 2nd kind, Eq. (3), in the spectral problem (19)–(21) the boundary condition (20) is replaced by the condition

$$\left.\frac{d\Theta(x,s)}{dx}\right|_{x=x_0} = 0\,. \tag{42}$$

The function $v(x, \lambda)$, connected with $\Theta(x, s)$ by relation (22), satisfies Eq. (23) with the condition

$$v'(x_0,\lambda) - \frac{m+1/2}{x_0}\,v(x_0,\lambda) = 0\,.$$

The functions $\chi(x, \lambda)$ and $\psi(x, \lambda)$ satisfy the conditions $\psi(x_0, \lambda) = 1$, $\psi'(x_0, \lambda) = (m + 1/2)x_0$, $\chi(x_0, \lambda) = 0$, $\chi'(x_0, \lambda) = -1$ and have the form

$$\psi(x,\lambda) = -\frac{2}{\pi}\,s\sqrt{xx_0}\,[J_m(sx)\,Y_{m+1}(sx_0) - Y_m(sx)\,J_{m+1}(sx_0)]\,,$$

$$\chi(x, \lambda) = \frac{2}{\pi} \sqrt{xx_0} \, [J_m(sx) \, Y_m(sx_0) - Y_m(sx) \, J_m(sx_0)] \, .$$

All the remaining considerations are repeated analogously. We will present the final results: the integral transformation

$$\bar{T}(s, t) = \int_{x_0}^{\infty} \Theta(x, s) \, T(x, t) \, x^{2m+1} \, dx \, ; \tag{43}$$

the kernel of the integral transformation

$$\Theta(x, s) = x^{-m} \, [J_m(sx) \, Y_{m+1}(sx_0) - Y_m(sx) \, J_{m+1}(sx_0)] \, ; \tag{44}$$

the image of the operator $\Delta T(x, t)$

$$\int_{x_0}^{\infty} \Theta(x, s) \, \Delta T(x, t) \, x^{2m+1} \, dx = \frac{2x_0^m}{\pi s} \frac{\partial T(x, t)}{\partial x} \bigg|_{x=x_0} - s^2 \bar{T}(s, t) \, ; \tag{45}$$

the conversion formula

$$T(x, t) = \int_{0}^{\infty} \frac{\Theta(x, s) \, \bar{T}(s, t)}{J_{m+1}^2(sx_0) + Y_{m+1}^2(sx_0)} \, sds \, . \tag{46}$$

As the illustration of relations (43)–(46) we will write the solution of problem (39)–(41), but with the boundary condition of the 2nd kind:

$$\frac{\partial T(x, t)}{\partial x} \bigg|_{x=x_0} = -\frac{1}{\lambda_t} q \, , \quad t > 0 \, . \tag{47}$$

Assuming in (43)–(46) that $m = 0$, we find

$$T(r, t) = -\frac{2q}{\pi \lambda_t} \int_{0}^{\infty} \frac{[1 - \exp(-as^2 t)] \, [J_0(sr) \, Y_1(sr_0) - Y_0(sr) \, J_1(sr_0)]}{s^2 \, [J_1^2(sr_0) + Y_1^2(sr_0)]} \, ds \, . \tag{48}$$

The operational solution of the second boundary-value problem (39)–(40), (42) has the form

$$\bar{T}(r, p) = \frac{q}{\lambda_t} \frac{K_0 \sqrt{p/(ar)}}{p/\sqrt{p/a} \, K_1 \sqrt{p/(ar_0)}}$$

and it is also connected with prolonged transformation in transition to the inverted transform.

For the boundary conditions of the 3rd kind (4) in the spectral problem (19)–(21) the boundary condition (20) is replaced by the condition

$$\left[\frac{d\Theta(x, s)}{dx} - h\Theta(x, s) \right] \bigg|_{x=x_0} = 0 \, . \tag{49}$$

The function $v(x, \lambda)$, connected with $\Theta(x, \lambda)$ by relation (22) satisfies Eq. (23) with the condition

$$v'(x_0, \lambda) - \left(\frac{m + 1/2}{x_0} - h\right) v(x_0, \lambda) = 0 ,$$

and the functions $\chi(x, \lambda)$ and $\psi(x, \lambda)$ satisfy the conditions $\psi(x_0, \lambda) = 1$, $\psi'(x_0, \lambda) = (m + 1/2)/x_0 + h$, $\chi(x_0, \lambda) = 0$, $\chi'(x_0, \lambda) = -1$ and have the form

$$\psi(x, \lambda) = -\frac{2}{\pi} s \sqrt{xx_0} \left[J_m(sx) Y_{m+1}(sx_0) - Y_m(sx) J_{m+1}(sx_0)\right] - h\chi(x, \lambda) ,$$

$$\chi(x, \lambda) = -\frac{2}{\pi} \sqrt{xx_0} \left[J_m(sx) Y_m(sx_0) - Y_m(sx) J_m(sx_0)\right] .$$

Further, all the considerations, just as in the case of Eq. (20), are repeated analogously. We will write out the final results: the integral transformation

$$\overline{T}(s, t) = \int_{x_0}^{\infty} \Theta(x, s) T(x, t) x^{2m+1} dx ; \tag{50}$$

the kernel of the integral transformation

$$\Theta(x, s) = x^{-m} \left\{\left[J_m(sx) Y_{m+1}(sx_0) - Y_m(sx) J_{m+1}(sx_0)\right] + \frac{h}{s}\left[J_m(sx) Y_m(sx_0) - \dot{Y}_m(sx) J_m(sx_0)\right]\right\} ; \tag{51}$$

the transformation of the operator $\Delta T(x, t)$

$$\int_{x_0}^{\infty} \Theta(x, s) \Delta T(x, t) x^{2m+1} dx = \frac{2x_0^m}{\pi s}\left[\frac{\partial T(x, t)}{\partial x} - hT(x, t)\right]\Bigg|_{x=x_0} - s^2 \overline{T}(s, t) ; \tag{52}$$

the conversion formula

$$T(x, t) = \int_0^{\infty} \frac{\theta(x, s) \overline{T}(s, t)}{J_{m+1}^2(sx_0) + Y_{m+1}^2(sx_0)} sds . \tag{53}$$

The transformations obtained are applicable for finding analytical solutions of a large number of boundary-value problems of nonstationary and stationary transfer in Cartesian, cylindrical, and spherical coordinate systems in an infinite region bounded on the inside by corresponding surfaces, and in contrast to the operational method — the basic approach in studying such kind of cases — lead rather rapidly to the goal with respect to the standard scheme reflected in the relations given.

NOTATION

a, thermal diffusivity; A, constant; $H_m^{(1)}(z)$, Hankel function of the 1st kind; $H_m^{(2)}(z)$, Hankel function of the 2nd kind; h, relative coefficient of heat transfer; J_m, Bessel function of the 1st kind; i, imaginary unit; $L^{(2)}$, class of functions; L, operator; p, parameter in the Laplace transformation; r, polar radius; r_0, boundary value of the polar radius; s, component of the argument; t, time; T, temperature; T_0, boundary temperature; v, function; x, z, arguments; x_0, boundary value of the argument; α_1, α_2, β_1, and β_2, coefficients; Δ, Laplace operator; λ, argument; λ_t, thermal conductivity; $\chi(x, \lambda)$ and $\psi(x, \lambda)$, functions.

REFERENCES

1. A. V. Luikov, *Heat Conduction Theory* [in Russian], Vysshaya Shkola, Moscow (1967).
2. N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov, *Partial Derivative Equations of Mathematical Physics* [in Russian], Vysshaya Shkola, Moscow (1970).
3. I. K. Volkov and A. N. Kanatnikov, *Integral Transformations and Operational Calculus* [in Russian], Izd. MGTU im. N. E. Baumana, Moscow (1996).
4. E. M. Kartashov, *Analytical Methods in the Heat-Conduction Theory of Solids* [in Russian], Vysshaya Shkola, Moscow (2001).
5. E. M. Kartashov, Method of integral transformations in the analytical heat-conduction theory of solids, *Izv. Ross. Akad. Nauk*, *Energetika*, No. 2, 99–127 (1993).
6. E. M. Kartashov, On improvement of the convergence of Fourier–Hankel series in the method of integral transformations, *Izv. Ross. Akad. Nauk*, *Energetika*, No. 3, 106–125 (1993).
7. M. A. Fatykhov and G. P. Smirnov, Toward the solution of a heat-conduction problem, *Differents. Uravn.*, **20**, No. 5, 899–901 (1984).